## Problem 4-1

The key point of this problem is to determine the positive values of K in the open-loop system that make the closed-loop system stable. This problem can be easily solved by using the root locus.

The open-loop transfer function provided in the problem is shown in Equation (1-1).

|  |  |
| --- | --- |
|  | (1-1) |

According to the graph in Problem 4-1, the closed-loop transfer function of the system can be written as Equation (1-2).

|  |  |
| --- | --- |
|  | (1-2) |

The characteristic equation, which is the denominator of Equation (1-2), can be written as Equation (1-3).

|  |  |
| --- | --- |
|  | (1-3) |

The problem also provides a constraint for K, which is indicated in Equation (1-4).

|  |  |
| --- | --- |
|  | (1-4) |

We can determine the stability of the closed-loop system by using the root locus. As in Equation (1-3), we can see that the starting points are places where z is equal to 0, 0.2, and, 0.4 and the crossing point of the asymptotes, which have directions in , is 0.2.

We then need to find the points on the asymptotes that intersect with the unit circle. Since the intersecting points are situated in a plane composed of real and imaginary values, we set the intersecting point as z in Equation (1-5).

|  |  |
| --- | --- |
|  | (1-5) |

Since the intersecting points land on the unit circle, we also know that

|  |  |
| --- | --- |
|  | (1-6) |

By substituting Equation (1-5) into Equation (1-3), we can get Equation (1-7). And by moving the terms unrelated to K to the right-hand side, we get Equation (1-8).

|  |  |
| --- | --- |
|  | (1-7) |
|  | (1-8) |

We then multiply both sides with  as in Equation (1-9).

|  |  |
| --- | --- |
|  | (1-9) |

From Equation (1-6), we know that , which is also equivalent to the multiplication of the first two terms on the right-hand side of Equation (1-9). Thus, by reordering the terms and substituting Equation (1-6) into Equation (1-9), we get Equation (1-10).

|  |  |
| --- | --- |
|  | (1-10) |

By expanding the terms in Equation (1-10), we get Equation (1-11).

|  |  |
| --- | --- |
|  | (1-11) |

By equaling the real and imaginary parts on both sides of Equation (1-11), we get Equation (1-12) and Equation (1-13).

|  |  |
| --- | --- |
|  | (1-12) |
|  | (1-13) |

If we assume that the value of y in Equation (1-12) is a number other than zero, we can express K in terms of x as in Equation (1-14).

|  |  |
| --- | --- |
|  | (1-14) |

By moving the terms of y on the left-hand side of Equation (1-6) to the right-hand side, we can express the square of y as in Equation (1-15).

|  |  |
| --- | --- |
|  | (1-15) |

We then get Equation (1-16) by substituting Equation (1-14) and Equation (1-15) into Equation (1-12), which leads us to solving the value of x.

|  |  |
| --- | --- |
|  | (1-16) |

The values of x indicated in Equation (1-17) can be obtained by solving Equation (1-16).

|  |  |
| --- | --- |
|  | (1-17) |

By substituting the solutions of into Equation (1-14), we solve the values of K, which are indicated in Equation (1-18) and Equation (1-19).

|  |  |
| --- | --- |
|  | (1-18) |
|  | (1-19) |

On the other hand, we obtain another pair of values for x, -1and 1, by considering y equal to zero. This is because y is zero whenever the unit circle intersects the real axis. By substituting x equal to -1 and 1 into Equation (1-10) respectively, we obtain two formulas, Equation (1-20) and Equation (1-21), for solving the other pair of values of K.

|  |  |
| --- | --- |
|  | (1-20) |
|  | (1-21) |

Equation (1-20) provides us with the solution K equal to 1.68, as shown in Equation (1-22), while Equation (1-21) concludes us with the value of K equal to -0.48, shown in Equation (1-23).

|  |  |
| --- | --- |
|  | (1-22) |
|  | (1-23) |

From Equation (1-18), Equation (1-19), Equation (1-22), Equation (1-23), and the constraint listed in Equation (1-4), we conclude that the values of K that make the closed-loop system stable range from 0 to 0.7. The answer is shown in Equation (1-24).

|  |  |
| --- | --- |
|  | (1-24)  (Ans.) |

## Problem 4-2

This problem aims to analyze whether the given system is stable, observable and reachable. To obtain the results, we first have to compute the observability matrix and the controllability matrix.

The problem provides us with the system shown in Equation (2-1) and Equation (2-2).

|  |  |
| --- | --- |
|  | (2-1) |
|  | (2-2) |

The system is stable if it satisfies the condition indicated in Equation (2-3).

|  |  |
| --- | --- |
|  | (2-3) |

It is easily seen that Equation (2-1) doesn’t satisfy Equation (2-3). Thus, the system is not stable.

To determine whether the system is observable and controllable, we have to compute the observability matrix and the controllability matrix. We first set the two matrices in Equation (2-1) as matrix F and matrix H respectively and the matrix in Equation (2-2) as matrix C. Equation (2-4), Equation (2-5), and Equation (2-6) show matrices F, H, and C respectively.

|  |  |
| --- | --- |
|  | (2-4) |
|  | (2-5) |
|  | (2-6) |

The observability matrix  can be expressed as Equation (2-7).

|  |  |
| --- | --- |
|  | (2-7) |

By substituting Equation (2-4) and Equation (2-6) into Equation (2-7), we obtain the result of the observability matrix, indicated in Equation (2-8).

|  |  |
| --- | --- |
|  | (2-8) |

It is easily seen that the rank of the observability matrix is one and the determinant of the observability matrix is zero, which means that the system is not observable.

As for controllability, we have to compute the controllability matrix first hand.

Since the system is a two-order system. the controllability matrix, , is expressed as Equation (2-9).

|  |  |
| --- | --- |
|  | (2-9) |

By substituting Equation (2-4) and Equation (2-5) into Equation (2-9), we compute the value of the controllability matrix as in Equation (2-10).

|  |  |
| --- | --- |
|  | (2-10) |

From Equation (2-10), we can see that the controllability matrix is full rank, which means that the system is reachable.

From above, we can conclude that the system in this problem is stable and reachable, but unobservable.

## Problem 4-3

In this problem, we have to first determine a control sequence for the given system such that the system is taken from the initial state,, to the origin, while also counting the minimum number of steps in order to determine the control sequence. Then, we have to explain why a given state cannot be reached from the origin with a sequence of control signals.

Part(a):

The problem considers the system mentioned in Equation (3-1), with the initial state given as in Equation (3-2).

|  |  |
| --- | --- |
|  | (3-1) |
|  | (3-2) |

To determine the control sequence of the given system, we have to compute the values of , with  equivalent to 0.

, the initial state, is provided in the problem as in Equation (3-2). We first compute the value of by substituting Equation (3-2) into Equation (3-1) and taking k as 0. Equation (3-3) shows the value of .

|  |  |
| --- | --- |
|  | (3-3) |

As for , we take k as 1 and substitute the value of into Equation (3-1). The result is shown in Equation (3-4).

|  |  |
| --- | --- |
|  | (3-4) |

Considering the case where  is zero, we can obtain the control sequence indicated in Equation (3-7), where the values of  and  are indicated in Equation (3-5) and Equation (3-6) respectively.

|  |  |
| --- | --- |
|  | (3-5) |
|  | (3-6) |
|  | (3-7)  (Ans.) |

Part (b):

Since the system is a third-order system, it would generally take us three steps to solve the problem. But as from the results in Part (a), we can see that the minimum number of steps to solve the problem is two.

Part (c):

To explain why it is not possible to find a sequence of control signals such that the state can be reached from the origin, we have to compute the controllability matrix.

The controllability matrix of the system is calculated as in Equation (3-8).

|  |  |
| --- | --- |
|  | (3-8) |

We can easily see that the initial state  isn’t equivalent to any of the columns in the controllability matrix. Thus, it is impossible for the state  to be reached from the origin. And since from Part (a), we already know that equals zero, which means that for any value of n larger than two, the value of  will be zero.

## Problem 4-4

The system provided in the problem is given as in Equation (4-1). For the given system, we have to find the values of K in a proportional controller that makes the closed-loop system stable and determine the stationary error under given conditions.

|  |  |
| --- | --- |
|  | (4-1) |

Part (a):

For part (a), we have to find the values of K in the proportional controller shown in Equation (4-2), such that the closed-loop system is stable.

|  |  |
| --- | --- |
|  | (4-2) |

By substituting Equation (4-2) into Equation (4-1), we get the following Equation (4-3).

|  |  |
| --- | --- |
|  | (4-3) |

We then get Equation (4-4) by moving the y terms to the left-hand side of Equation (4-3). We then see that the transfer operator of the system as in Equation (4-5).

|  |  |
| --- | --- |
|  | (4-4) |
|  | (4-5) |

To determine the values of K that make the closed-loop system to be stable, we apply Jury’s Stability Criterion. The characteristic equation is indicated in Equation (4-6).

|  |  |
| --- | --- |
|  | (4-6) |

Jury’s scheme can then be expressed as in Equation (4-7).

|  |  |
| --- | --- |
|  | (4-7) |

To have a stable closed-loop system, the roots should be situated inside the unit circle. This leads us to Equation (4-8) and Equation (4-9).

|  |  |
| --- | --- |
|  | (4-8) |
|  | (4-9) |

By solving Equation (4.8), we obtain the first constraint for K, which is indicated in Equation (4-10).

|  |  |
| --- | --- |
|  | (4-10) |

And by solving Equation (4-9), we come up with the other two constraints of K, being Equation (4-11) and Equation (4-12).

|  |  |
| --- | --- |
|  | (4-11) |
|  | (4-12) |

From Equation (4-10), Equation (4-11), and Equation (4-12), we can conclude that for the closed-loop system to be stable, the value of K has to be in the range between -0.6 and 1. Equation (4-13) shows the final answer of Part (a).

|  |  |
| --- | --- |
|  | (4-13)  (Ans.) |

Part (b):

The stationary error can be rewritten as a function of K, as shown in Equation (4-14).

|  |  |
| --- | --- |
|  | (4-14) |

By substituting Equation (4-4) into Equation (4-14), we can express the stationary error as in Equation (4-15).

|  |  |
| --- | --- |
|  | (4-15) |

From the problem in part (b), we can see that is set as a step function and K is set as 0.5, which lies in the range of values such that it makes the closed-loop system stable. Thus, we can apply the final-value theorem to solve this problem. By applying the final-value theorem, the stationary error is then written as in Equation (4-16).

|  |  |
| --- | --- |
|  | (4-16) |

By substituting Equation (4-15) into Equation (4-16) and considering as a step function, we then get Equation (4-17).

|  |  |
| --- | --- |
|  | (4-17) |

We can obtain the result of Equation (4-17) if we take the limit of z as 1 and substitute K with 0.5. The result is indicated in Equation (4-18).

|  |  |
| --- | --- |
|  | (4-18) |

We then conclude that the stationary error of the system is 0.74. Equation (4-19) shows the final answer to Part (b).

|  |  |
| --- | --- |
|  | (4-19)  (Ans.) |